The boundedness of some singular integral operators on weighted Hardy spaces associated with Schrödinger operators

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Abstract

Let $L = -\Delta + V$ be a Schrödinger operator acting on $L^2(\mathbb{R}^n)$, $n \geq 1$, where $V \not\equiv 0$ is a nonnegative locally integrable function on $\mathbb{R}^n$. In this paper, we first define molecules for weighted Hardy spaces $H^p_L(w)(0 < p \leq 1)$ associated to L and establish their molecular characterizations. Then by using the atomic decomposition and molecular characterization of $H^p_L(w)$, we will show that the imaginary power $L^{i\gamma}$ is bounded on $H^p_L(w)$ for $n/(n+1) < p \leq 1$, and the fractional integral operator $L^{-\alpha/2}$ is bounded from $H^p_L(w)$ to $H^q_L(w^{q/p})$, where $0 < \alpha < \min\{n/2, 1\}$, $n/(n+1) < p \leq n/(n+\alpha)$ and $1/q = 1/p - \alpha/n$.

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1 Introduction

Let $n \geq 1$ and V be a nonnegative locally integrable function defined on $\mathbb{R}^n$, not identically zero. We define the form $\mathcal{Q}$ by

$$\mathcal{Q}(u,v) = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^n} V uv \, dx$$

with domain $\mathcal{D}(\mathcal{Q}) = \mathcal{V} \times \mathcal{V}$ where

$$\mathcal{V} = \{u \in L^2(\mathbb{R}^n) : \frac{\partial u}{\partial x_k} \in L^2(\mathbb{R}^n) \text{ for } k = 1, \dots, n \text{ and } \sqrt{V} u \in L^2(\mathbb{R}^n)\}.$$

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It is well known that this symmetric form is closed. Note also that it was shown by Simon [15] that this form coincides with the minimal closure of the form given by the same expression but defined on $C_0^\infty(\mathbb{R}^n)$(the space of C^∞ functions with compact supports). In other words, $C_0^\infty(\mathbb{R}^n)$ is a core of the form $\mathcal{Q}$.

Let us denote by L the self-adjoint operator associated with $\mathcal{Q}$. The domain of L is given by

$$\mathcal{D}(L) = \{u \in \mathcal{D}(\mathcal{Q}) : \exists\, v \in L^2 \text{ such that } \mathcal{Q}(u,\varphi) = \int_{\mathbb{R}^n} v\varphi\, dx, \forall \varphi \in \mathcal{D}(\mathcal{Q})\}.$$

Formally, we write $L = -\Delta + V$ as a Schrödinger operator with potential V. Let $\{e^{-tL}\}_{t>0}$ be the semigroup of linear operators generated by $-L$ and $p_t(x,y)$ be their kernels. Since V is nonnegative, the Feynman-Kac formula implies that

$$0 \le p_t(x,y) \le \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} \tag{1.1}$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$.

Since the Schrödinger operator L is a self-adjoint positive definite operator acting on $L^2(\mathbb{R}^n)$, then L admits the following spectral resolution

$$L = \int_0^\infty \lambda\, dE_L(\lambda),$$

where the $E_L(\lambda)$ are spectral projectors. For any $\gamma \in \mathbb{R}$, we shall define the imaginary power $L^{i\gamma}$ associated to L by the formula

$$L^{i\gamma} = \int_0^\infty \lambda^{i\gamma}\, dE_L(\lambda).$$

By the functional calculus for L, we can also define the operator $L^{i\gamma}$ as follows

$$L^{i\gamma}(f)(x) = \frac{1}{\Gamma(-i\gamma)} \int_0^\infty t^{-i\gamma-1} e^{-tL}(f)(x)\, dt. \tag{1.2}$$

By spectral theory $\|L^{i\gamma}\|_{L^2 \to L^2} = 1$ for all $\gamma \in \mathbb{R}$. Moreover, it was proved by Shen [12] that $L^{i\gamma}$ is a Calderón-Zygmund operator provided that $V \in RH_{n/2}$(Reverse Hölder class). We refer the readers to [6,7,14] for related results concerning the imaginary powers of self-adjoint operators.

For any $0 < \alpha < n$, the fractional integrals $L^{-\alpha/2}$ associated to L is defined by

$$L^{-\alpha/2}(f)(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} e^{-tL}(f)(x)\, dt. \tag{1.3}$$

Since the kernel $p_t(x,y)$ of $\{e^{-tL}\}_{t>0}$ satisfies the Gaussian upper bound (1.1), then it is easy to check that $\left|L^{-\alpha/2}(f)(x)\right| \le CI_\alpha(|f|)(x)$ for all $x \in \mathbb{R}^n$, where I_α denotes the classical fractional integral operator(see [17])

$$I_\alpha(f)(x) = \frac{\Gamma(\frac{n-\alpha}{2})}{2^\alpha \pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

Hence, by using the L^p-L^q boundedness of I_α(see [17]), we have

$$\|L^{-\alpha/2}(f)\|_{L^q} \le C\|I_\alpha(f)\|_{L^q} \le C\|f\|_{L^p},$$

where $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. For more information about the fractional integrals $L^{-\alpha/2}$ associated to a general class of operators, we refer the readers to [3,9,20].

In [16], Song and Yan introduced the weighted Hardy spaces $H^1_L(w)$ associated to L in terms of the area integral function and established their atomic decomposition theory. They also showed that the Riesz transform $\nabla L^{-1/2}$ is bounded on $L^p(w)$ for $1 < p < 2$, and bounded from $H^1_L(w)$ to the classical weighted Hardy space $H^1(w)$(see [4,18]).

Recently, in [19], we defined the weighted Hardy spaces $H^p_L(w)$ associated to L for $0 < p < 1$ and gave their atomic decompositions. We also obtained that $\nabla L^{-1/2}$ is bounded from $H^p_L(w)$ to the classical weighted Hardy space $H^p(w)$(see also [4,18]) for $n/(n+1) < p < 1$. In this article, we first define weighted molecules for the weighted Hardy spaces $H^p_L(w)$ associated to L and then establish their molecular characterizations. As an application of the molecular characterization combining with the atomic decomposition of $H^p_L(w)$, we shall obtain some estimates of $L^{i\gamma}$ and $L^{-\alpha/2}$ on $H^p_L(w)$ for $n/(n+1) < p \le 1$. Our main results are stated as follows.

Theorem 1.1. *Let $L = -\Delta + V$, $n/(n+1) < p \le 1$ and $w \in A_1 \cap RH_{(2/p)'}$. Then for any $\gamma \in \mathbb{R}$, the imaginary power $L^{i\gamma}$ is bounded from $H^p_L(w)$ to the weighted Lebesgue space $L^p(w)$.*

Theorem 1.2. *Let $L = -\Delta + V$, $n/(n+1) < p \le 1$ and $w \in A_1 \cap RH_{(2/p)'}$. Then for any $\gamma \in \mathbb{R}$, the imaginary power $L^{i\gamma}$ is bounded on $H^p_L(w)$.*

Theorem 1.3. *Suppose that $L = -\Delta + V$. Let $0 < \alpha < n/2$, $n/(n+1) < p \le 1$, $1/q = 1/p - \alpha/n$ and $w \in A_1 \cap RH_{(2/p)'}$. Then the fractional integral operator $L^{-\alpha/2}$ is bounded from $H^p_L(w)$ to $L^q(w^{q/p})$.*

Theorem 1.4. *Suppose that $L = -\Delta + V$. Let $0 < \alpha < \min\{n/2, 1\}$, $n/(n+1) < p \le n/(n+\alpha)$, $1/q = 1/p - \alpha/n$ and $w \in A_1 \cap RH_{(2/p)'}$. Then the fractional integral operator $L^{-\alpha/2}$ is bounded from $H^p_L(w)$ to $H^q_L(w^{q/p})$.*

2 Notations and preliminaries

First, let us recall some standard definitions and notations. The classical A_p weight theory was first introduced by Muckenhoupt in the study of weighted L^p boundedness of Hardy-Littlewood maximal functions in [10]. A weight w is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ almost everywhere, $B = B(x_0, r_B)$ denotes the ball with the center x_0 and radius r_B. We say that $w \in A_1$, if

$$\frac{1}{|B|}\int_B w(x)\,dx \le C \cdot \operatorname*{ess\,inf}_{x\in B} w(x) \quad \text{for every ball } B \subseteq \mathbb{R}^n.$$

where C is a positive constant which is independent of B.

A weight function w is said to belong to the reverse Hölder class RH_r if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality holds

$$\left(\frac{1}{|B|}\int_B w(x)^r\,dx\right)^{1/r} \le C\left(\frac{1}{|B|}\int_B w(x)\,dx\right) \quad \text{for every ball } B \subseteq \mathbb{R}^n.$$

Given a ball B and $\lambda > 0$, λB denotes the ball with the same center as B whose radius is λ times that of B. For a given weight function w, we denote the Lebesgue measure of B by $|B|$ and the weighted measure of B by $w(B)$, where $w(B) = \int_B w(x)\,dx$.

We give the following results which will be often used in the sequel.

Lemma 2.1 ([5]). *Let $w \in A_1$. Then, for any ball B, there exists an absolute constant C such that*

$$w(2B) \le C\,w(B).$$

In general, for any $\lambda > 1$, we have

$$w(\lambda B) \le C \cdot \lambda^n w(B),$$

where C does not depend on B nor on λ.

Lemma 2.2 ([5]). *Let $w \in A_1$. Then there exists a constant $C > 0$ such that*

$$C \cdot \frac{|E|}{|B|} \le \frac{w(E)}{w(B)}$$

for any measurable subset E of a ball B.

Given a Muckenhoupt's weight function w on $\mathbb{R}^n$, for $0 < p < \infty$, we denote by $L^p(w)$ the space of all functions satisfying

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx \right)^{1/p} < \infty.$$

Throughout this article, we will use C to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \le \frac{A}{B} \le C$. Moreover, we denote the conjugate exponent of $s > 1$ by $s' = s/(s-1)$.

3 Atomic decomposition and molecular characterization of weighted Hardy spaces

Let $L = -\Delta + V$. For any $t > 0$, we define $P_t = e^{-tL}$ and

$$Q_{t,k} = (-t)^k \left. \frac{d^k P_s}{ds^k} \right|_{s=t} = (tL)^k e^{-tL}, \quad k = 1, 2, \dots.$$

We denote simply by Q_t when $k = 1$. First note that Gaussian upper bounds carry over from heat kernels to their time derivatives.

Lemma 3.1 ([2,11])**.** *For every $k = 1, 2, \dots,$ there exist two positive constants C_k and c_k such that the kernel $p_{t,k}(x, y)$ of the operator $Q_{t,k}$ satisfies*

$$\left|p_{t,k}(x, y)\right| \le \frac{C_k}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{c_k t}}$$

for all $t > 0$ and almost all $x, y \in \mathbb{R}^n$.

Set

$$H^2(\mathbb{R}^n) = \overline{\mathcal{R}(L)} = \overline{\{Lu \in L^2(\mathbb{R}^n) : u \in L^2(\mathbb{R}^n)\}},$$

where $\overline{\mathcal{R}(L)}$ stands for the range of L. We also set

$$\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ : |x - y| < t\}.$$

For a given function $f \in L^2(\mathbb{R}^n)$, we consider the area integral function associated to Schrödinger operator L(see [1,8])

$$S_L(f)(x) = \left(\iint_{\Gamma(x)} \left|Q_{t^2}(f)(y)\right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \quad x \in \mathbb{R}^n.$$

Given a weight function w on $\mathbb{R}^n$, in [16,19], the authors defined the weighted Hardy spaces $H^p_L(w)$ for $0 < p \le 1$ as the completion of $H^2(\mathbb{R}^n)$ in the norm given by the $L^p(w)$-norm of area integral function; that is

$$\|f\|_{H^p_L(w)} = \|S_L(f)\|_{L^p(w)}.$$

In [16], Song and Yan characterized weighted Hardy spaces $H^1_L(w)$ in terms of atoms in the following way and obtained their atomic characterizations.

Definition 3.2 ([16]). *Let $M \in \mathbb{N}$. A function $a(x) \in L^2(\mathbb{R}^n)$ is called a $(1,2,M)$-atom with respect to w(or a w-$(1,2,M)$-atom) if there exist a ball $B = B(x_0, r_B)$ and a function $b \in \mathcal{D}(L^M)$ such that*

(a) $a = L^M b$;

(b) $\operatorname{supp} L^k b \subseteq B, \quad k = 0,1,\dots,M$;

(c) $\|(r_B^2 L)^k b\|_{L^2(B)} \le r_B^{2M} |B|^{1/2} w(B)^{-1}, \quad k = 0,1,\dots,M$.

Theorem 3.3 ([16]). *Let $M \in \mathbb{N}$ and $w \in A_1 \cap RH_2$. If $f \in H^1_L(w)$, then there exist a family of w-$(1,2,M)$-atoms $\{a_j\}$ and a sequence of real numbers $\{\lambda_j\}$ with $\sum_j |\lambda_j| \le C\|f\|_{H^1_L(w)}$ such that f can be represented in the form $f(x) = \sum_j \lambda_j a_j(x)$, and the sum converges both in the sense of $L^2(\mathbb{R}^n)$-norm and $H^1_L(w)$-norm.*

Similarly, in [19], we introduced the notion of weighted atoms for $H^p_L(w)(0 < p < 1)$ and proved their atomic characterizations.

Definition 3.4 ([19]). *Let $M \in \mathbb{N}$ and $0 < p < 1$. A function $a(x) \in L^2(\mathbb{R}^n)$ is called a $(p,2,M)$-atom with respect to w(or a w-$(p,2,M)$-atom) if there exist a ball $B = B(x_0, r_B)$ and a function $b \in \mathcal{D}(L^M)$ such that*

(a′) $a = L^M b$;

(b′) $\operatorname{supp} L^k b \subseteq B, \quad k = 0,1,\dots,M$;

(c′) $\|(r_B^2 L)^k b\|_{L^2(B)} \le r_B^{2M} |B|^{1/2} w(B)^{-1/p}, \quad k = 0,1,\dots,M$.

Theorem 3.5 ([19]). *Let $M \in \mathbb{N}$, $0 < p < 1$ and $w \in A_1 \cap RH_{(2/p)'}$. If $f \in H^p_L(w)$, then there exist a family of w-$(p,2,M)$-atoms $\{a_j\}$ and a sequence of real numbers $\{\lambda_j\}$ with $\sum_j |\lambda_j|^p \le C\|f\|^p_{H^p_L(w)}$ such that f can be represented in the form $f(x) = \sum_j \lambda_j a_j(x)$, and the sum converges both in the sense of $L^2(\mathbb{R}^n)$-norm and $H^p_L(w)$-norm.*

For every bounded Borel function $F : [0,\infty) \to \mathbb{C}$, we define the operator $F(L) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by the following formula

$$F(L) = \int_0^\infty F(\lambda)\, dE_L(\lambda),$$

where $E_L(\lambda)$ is the spectral decomposition of L. Therefore, the operator $\cos(t\sqrt{L})$ is well-defined on $L^2(\mathbb{R}^n)$. Moreover, it follows from [13] that there exists a constant c_0 such that the Schwartz kernel $K_{\cos(t\sqrt{L})}(x,y)$ of $\cos(t\sqrt{L})$ has support contained in $\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : |x-y|\le c_0t\}$. By the functional calculus for L and Fourier inversion formula, whenever F is an even bounded Borel function with $\hat{F}\in L^1(\mathbb{R})$, we can write $F(\sqrt{L})$ in terms of $\cos(t\sqrt{L})$; precisely

$$F(\sqrt{L})=(2\pi)^{-1}\int_{-\infty}^{\infty}\hat{F}(t)\cos(t\sqrt{L})\,dt,$$

which gives

$$K_{F(\sqrt{L})}(x,y)=(2\pi)^{-1}\int_{|t|\ge c_0^{-1}|x-y|}\hat{F}(t)K_{\cos(t\sqrt{L})}(x,y)\,dt.$$

Lemma 3.6 ([8])**.** *Let $\varphi\in C_0^\infty(\mathbb{R})$ be even and supp $\varphi\subseteq[-c_0^{-1},c_0^{-1}]$. Let Φ denote the Fourier transform of φ. Then for each $j=0,1,\ldots,$ and for all $t>0$, the Schwartz kernel $K_{(t^2L)^j\Phi(t\sqrt{L})}(x,y)$ of $(t^2L)^j\Phi(t\sqrt{L})$ satisfies*

$$\mathit{supp}\,K_{(t^2L)^j\Phi(t\sqrt{L})}\subseteq\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n : |x-y|\le t\}.$$

For a given $s>0$, we set

$$\mathcal{F}(s)=\Big\{\psi:\mathbb{C}\to\mathbb{C}\text{ measurable}, |\psi(z)|\le C\frac{|z|^s}{1+|z|^{2s}}\Big\}.$$

Then for any nonzero function $\psi\in\mathcal{F}(s)$, we have the following estimate(see [16])

$$\Big(\int_0^\infty\|\psi(t\sqrt{L})f\|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t}\Big)^{1/2}\le C\|f\|_{L^2(\mathbb{R}^n)}. \tag{3.1}$$

We are now going to define the weighted molecules corresponding to the weighted atoms mentioned above.

Definition 3.7. *Let $\varepsilon>0$, $M\in\mathbb{N}$ and $0<p\le1$. A function $m(x)\in L^2(\mathbb{R}^n)$ is called a w-$(p,2,M,\varepsilon)$-molecule associated to L if there exist a ball $B=B(x_0,r_B)$ and a function $b\in\mathcal{D}(L^M)$ such that*

(A) $m=L^Mb$;

(B) $\|(r_B^2L)^kb\|_{L^2(2B)}\le r_B^{2M}|B|^{1/2}w(B)^{-1/p},\quad k=0,1,\ldots,M;$

(C) $\|(r_B^2L)^kb\|_{L^2(2^{j+1}B\backslash 2^jB)}\le 2^{-j\varepsilon}r_B^{2M}|2^jB|^{1/2}w(2^jB)^{-1/p}$, $k=0,1,\ldots,M,\quad j=1,2,\ldots.$

Note that for every w-$(p, 2, M)$-atom a, it is a w-$(p, 2, M, \varepsilon)$-molecule for all $\varepsilon > 0$. Then we are able to establish the following molecular characterization for the weighted Hardy spaces $H^p_L(w)(0 < p \le 1)$ associated to L.

Theorem 3.8. *Let $\varepsilon > 0$, $M \in \mathbb{N}$, $0 < p \le 1$ and $w \in A_1 \cap RH_{(2/p)'}$.*
(i) If $f \in H^p_L(w)$, then there exist a family of w-$(p, 2, M, \varepsilon)$-molecules $\{m_j\}$ and a sequence of real numbers $\{\lambda_j\}$ with $\sum_j |\lambda_j|^p \le C\|f\|^p_{H^p_L(w)}$ such that $f(x) = \sum_j \lambda_j m_j(x)$, and the sum converges both in the sense of $L^2(\mathbb{R}^n)$-norm and $H^p_L(w)$-norm.
(ii) Assume that $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$. Then every w-$(p, 2, M, \varepsilon)$-molecule m is in $H^p_L(w)$. Moreover, there exists a constant $C > 0$ independent of m such that $\|m\|_{H^p_L(w)} \le C$.

Proof. (i) is a straightforward consequence of Theorems 3.3 and 3.5.

(ii) We follow the same constructions as in [8]. Suppose that m is a w-$(p, 2, M, \varepsilon)$-molecule associated to a ball $B = B(x_0, r_B)$. Let $\varphi \in C^\infty_0(\mathbb{R})$ be even with $supp\,\varphi \subseteq [-(2c_0)^{-1}, (2c_0)^{-1}]$ and let Φ denote the Fourier transform of φ. We set $\Psi(x) = x^2\Phi(x), x \in \mathbb{R}$. By the L^2-functional calculus of L, for every $m \in L^2(\mathbb{R}^n)$, we can establish the following version of the Calderón reproducing formula

$$m(x) = c_\psi \int_0^\infty (t^2L)^M \Psi^2(t\sqrt{L})(m)(x)\frac{dt}{t}, \tag{3.2}$$

where the above equality holds in the sense of $L^2(\mathbb{R}^n)$-norm. Set $U_0(B) = 2B$, $U_j(B) = 2^{j+1}B\backslash 2^jB, j = 1, 2, \ldots$, then we can decompose

$$\mathbb{R}^n \times (0, \infty) = \Big(\bigcup_{j=0}^\infty U_j(B) \times (0, 2^jr_B]\Big) \bigcup \Big(\bigcup_{j=1}^\infty 2^jB \times (2^{j-1}r_B, 2^jr_B]\Big).$$

Hence, by the formula (3.2), we are able to write

$$\begin{aligned} m(x) =& c_\psi \sum_{j=0}^\infty \int_0^{2^jr_B} (t^2L)^M \Psi^2(t\sqrt{L})(m\chi_{U_j(B)})(x)\frac{dt}{t} \\ &+ c_\psi \sum_{j=1}^\infty \int_{2^{j-1}r_B}^{2^jr_B} (t^2L)^M \Psi^2(t\sqrt{L})(m\chi_{2^jB})(x)\frac{dt}{t} \\ =& \sum_{j=0}^\infty m^{(1)}_j(x) + \sum_{j=1}^\infty m^{(2)}_j(x). \end{aligned}$$

Let us first estimate the terms $\{m_j^{(1)}\}_{j=0}^{\infty}$. We will show that each $m_j^{(1)}$ is a multiple of w-$(p,2,M)$-atom with a sequence of coefficients in l^p. Indeed, for every $j=0,1,2,\ldots$, one can write

$$m_j^{(1)}(x) = L^M b_j(x),$$

where

$$b_j(x) = c_\psi \int_0^{2^j r_B} t^{2M}\Psi^2(t\sqrt{L})(m\chi_{U_j(B)})(x)\frac{dt}{t}.$$

By Lemma 3.6, we can easily conclude that for every $k=0,1,\ldots,M$, $supp\,(L^k b_j)\subseteq 2^{j+1}B$. Since

$$\left\|\left[(2^{j+1}r_B)^2L\right]^k b_j\right\|_{L^2(2^{j+1}B)} = \sup_{\|h\|_{L^2(2^{j+1}B)}\le 1}\left|\int_{2^{j+1}B}\left[(2^{j+1}r_B)^2L\right]^k b_j(x)h(x)\,dx\right|.$$

Then it follows from Hölder's inequality and the estimate (3.1) that

$$\begin{aligned}
&\left|\int_{2^{j+1}B}\left[(2^{j+1}r_B)^2L\right]^k b_j(x)h(x)\,dx\right|\\
=&c_\psi(2^{j+1}r_B)^{2k}\left|\int_0^{2^j r_B}\int_{2^{j+1}B} t^{2M}L^k\Psi(t\sqrt{L})(m\chi_{U_j(B)})(y)\Psi(t\sqrt{L})(h)(y)\frac{dydt}{t}\right|\\
\le&c_\psi(2^{j+1}r_B)^{2k}(2^j r_B)^{2M-2k}\left|\int_0^{2^j r_B}\int_{2^{j+1}B}(t^2L)^k\Psi(t\sqrt{L})(m\chi_{U_j(B)})(y)\Psi(t\sqrt{L})(h)(y)\frac{dydt}{t}\right|\\
\le&c_\psi(2^{j+1}r_B)^{2M}\left(\int_0^\infty\left\|(t^2L)^k\Psi(t\sqrt{L})(m\chi_{U_j(B)})\right\|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t}\right)^{1/2}\\
&\times\left(\int_0^\infty\left\|\Psi(t\sqrt{L})(h\chi_{2^{j+1}B})\right\|^2_{L^2(\mathbb{R}^n)}\frac{dt}{t}\right)^{1/2}\\
\le&c_\psi(2^{j+1}r_B)^{2M}\left\|m\chi_{U_j(B)}\right\|_{L^2(\mathbb{R}^n)}\cdot\left\|h\chi_{2^{j+1}B}\right\|_{L^2(\mathbb{R}^n)}\\
\le&C\cdot 2^{-j\varepsilon}(2^{j+1}r_B)^{2M}|2^{j+1}B|^{1/2}w(2^{j+1}B)^{-1/p}.
\end{aligned}$$

Hence

$$\left\|\left[(2^{j+1}r_B)^2L\right]^k b_j\right\|_{L^2(2^{j+1}B)}\le C\cdot 2^{-j\varepsilon}(2^{j+1}r_B)^{2M}|2^{j+1}B|^{1/2}w(2^{j+1}B)^{-1/p},$$

which implies our desired result. Next we consider the terms $\{m_j^{(2)}\}_{j=1}^{\infty}$.

For every $j = 1, 2, \ldots$, we write

$$\begin{aligned} m_j^{(2)}(x) =& c_\psi \int_{2^{j-1}r_B}^{2^j r_B} (t^2L)^M \Psi^2(t\sqrt{L})(m)(x)\frac{dt}{t} \\ &- c_\psi \int_{2^{j-1}r_B}^{2^j r_B} (t^2L)^M \Psi^2(t\sqrt{L})(m\chi_{(2^jB)^c})(x)\frac{dt}{t} \\ =& m_j^{(21)}(x) - m_j^{(22)}(x). \end{aligned}$$

To deal with the term $m_j^{(21)}$, we recall that $m = L^M b$ for some $b \in \mathcal{D}(L^M)$. Then we have

$$\begin{aligned} m_j^{(21)}(x) &= c_\psi \int_{2^{j-1}r_B}^{2^j r_B} (t^2L)^M \Psi^2(t\sqrt{L})(L^M b)(x)\frac{dt}{t} \\ &= L^M b_j^{(21)}(x), \end{aligned}$$

where

$$b_j^{(21)}(x) = c_\psi \int_{2^{j-1}r_B}^{2^j r_B} (t^2L)^M \Psi^2(t\sqrt{L})(b)(x)\frac{dt}{t}.$$

Since $b(x) = b(x)\chi_{2^jB}(x) + \sum_{l=j}^{\infty} b(x)\chi_{U_l(B)}(x)$. Then we can further write

$$b_j^{(21)}(x) = b_{1,j}^{(21)}(x) + \sum_{l=j}^{\infty} b_{lj}^{(21)}(x),$$

where

$$b_{1,j}^{(21)}(x) = c_\psi \int_{2^{j-1}r_B}^{2^j r_B} (t^2L)^M \Psi^2(t\sqrt{L})(b\chi_{2^jB})(x)\frac{dt}{t}$$

and

$$b_{lj}^{(21)}(x) = c_\psi \int_{2^{j-1}r_B}^{2^j r_B} (t^2L)^M \Psi^2(t\sqrt{L})(b\chi_{U_l(B)})(x)\frac{dt}{t}.$$

By using Lemma 3.6 again, we have *supp* $(L^k b_{1,j}^{(21)}) \subseteq 2^jB$ and *supp* $(L^k b_{lj}^{(21)}) \subseteq 2^{l+1}B$ for every $k = 0, 1, \ldots, M$. Moreover, it follows from Minkowski's integral inequality that

$$\begin{aligned} &\Big\| \big[(2^j r_B)^2 L\big]^k b_{1,j}^{(21)} \Big\|_{L^2(2^jB)} \\ =& c_\psi (2^j r_B)^{2k} \Big\| \int_{2^{j-1}r_B}^{2^j r_B} t^{2M} L^{M+k} \Psi^2(t\sqrt{L})(b\chi_{2^jB})\frac{dt}{t} \Big\|_{L^2(2^jB)} \\ \le& c_\psi (2^j r_B)^{2k} \int_{2^{j-1}r_B}^{2^j r_B} \big\|(t^2L)^{M+k}\Psi^2(t\sqrt{L})(b\chi_{2^jB})\big\|_{L^2(2^jB)}\frac{dt}{t^{2k+1}} \\ \le& C\big\|b\chi_{2^jB}\big\|_{L^2(2^jB)} \end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{l=0}^{j-1}\left\|b\chi_{U_l(B)}\right\|_{L^2(2^jB)}\\
&\leq C\sum_{l=0}^{j-1}2^{-l\varepsilon}r_B^{2M}|2^lB|^{1/2}w(2^lB)^{-1/p}.
\end{aligned}
$$

When $0\leq l\leq j-1$, then $2^lB\subseteq 2^jB$. By using Lemma 2.2, we can get

$$
\frac{w(2^lB)}{w(2^jB)}\geq C\cdot\frac{|2^lB|}{|2^jB|}.
$$

Consequently

$$
\begin{aligned}
&\left\|\left[(2^jr_B)^2L\right]^k b_{1,j}^{(21)}\right\|_{L^2(2^jB)}\\
&\leq C\cdot 2^{-j[2M-n(1/p-1/2)]}\cdot(2^jr_B)^{2M}|2^jB|^{1/2}w(2^jB)^{-1/p}\sum_{l=0}^{\infty}\frac{1}{2^{l\varepsilon}}\cdot\frac{1}{2^{l(n/p-n/2)}}\\
&\leq C\cdot 2^{-j[2M-n(1/p-1/2)]}\cdot(2^jr_B)^{2M}|2^jB|^{1/2}w(2^jB)^{-1/p}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\left\|\left[(2^{l+1}r_B)^2L\right]^k b_{lj}^{(21)}\right\|_{L^2(2^{l+1}B)}\\
&=c_\psi(2^{l+1}r_B)^{2k}\left\|\int_{2^{j-1}r_B}^{2^jr_B}t^{2M}L^{M+k}\Psi^2(t\sqrt{L})(b\chi_{U_l(B)})\frac{dt}{t}\right\|_{L^2(2^{l+1}B)}\\
&\leq c_\psi(2^{l+1}r_B)^{2k}\int_{2^{j-1}r_B}^{2^jr_B}\left\|(t^2L)^{M+k}\Psi^2(t\sqrt{L})(b\chi_{U_l(B)})\right\|_{L^2(2^{l+1}B)}\frac{dt}{t^{2k+1}}\\
&\leq C(2^{l+1}r_B)^{2k}\left\|b\chi_{U_l(B)}\right\|_{L^2(2^{l+1}B)}\cdot\frac{1}{(2^jr_B)^{2k}}\\
&\leq C\cdot 2^{-l\varepsilon}(2^{l+1}r_B)^{2M}|2^{l+1}B|^{1/2}w(2^{l+1}B)^{-1/p}.
\end{aligned}
$$

Observe that $2M>n(1/p-1/2)$. Thus, from the above discussions, we have proved that each $m_j^{(21)}$ is a multiple of w-$(p,2,M)$-atom with a sequence of coefficients in l^p. Finally, we estimate the terms $\left\{m_j^{(22)}\right\}_{j=1}^{\infty}$. For every $j=1,2,\dots,$ we decompose $m_j^{(22)}$ as follows

$$
\begin{aligned}
m_j^{(22)}(x)&=c_\psi\sum_{l=j}^{\infty}\int_{2^{j-1}r_B}^{2^jr_B}(t^2L)^M\Psi^2(t\sqrt{L})(m\chi_{U_l(B)})(x)\frac{dt}{t}\\
&=\sum_{l=j}^{\infty}L^Mb_{lj}^{(22)}(x),
\end{aligned}
$$

where

$$b_{lj}^{(22)}(x)=c_\psi\int_{2^{j-1}r_B}^{2^j r_B} t^{2M}\Psi^2(t\sqrt{L})(m\chi_{U_l(B)})(x)\frac{dt}{t}.$$

It follows immediately from Lemma 3.6 that $supp\,(L^k b_{lj}^{(22)})\subseteq 2^{l+1}B$ for every $k=1,2,\dots,M$ and $l\geq j$. Moreover

$$\begin{aligned}
&\Big\|\big[(2^{l+1}r_B)^2L\big]^k b_{lj}^{(22)}\Big\|_{L^2(2^{l+1}B)}\\
=&c_\psi(2^{l+1}r_B)^{2k}\Big\|\int_{2^{j-1}r_B}^{2^j r_B} t^{2M}L^k\Psi^2(t\sqrt{L})(m\chi_{U_l(B)})\frac{dt}{t}\Big\|_{L^2(2^{l+1}B)}\\
\leq&c_\psi(2^{l+1}r_B)^{2k}(2^l r_B)^{2M-2k}\int_{2^{j-1}r_B}^{2^j r_B}\big\|(t^2L)^k\Psi^2(t\sqrt{L})(m\chi_{U_l(B)})\big\|_{L^2(2^{l+1}B)}\frac{dt}{t}\\
\leq&c_\psi(2^{l+1}r_B)^{2M}\big\|m\chi_{U_l(B)}\big\|_{L^2(2^{l+1}B)}\\
\leq&C\cdot 2^{-l\varepsilon}(2^{l+1}r_B)^{2M}|2^{l+1}B|^{1/2}w(2^{l+1}B)^{-1/p}.
\end{aligned}$$

Therefore, we have showed that each $m_j^{(22)}$ is also a multiple of w-$(p,2,M)$-atom with a sequence of coefficients in l^p. This completes the proof of Theorem 3.8. □

4 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. For any $\gamma\in\mathbb{R}$, since the operator $L^{i\gamma}$ is linear and bounded on $L^2(\mathbb{R}^n)$, then by Theorems 3.3 and 3.5, it is enough to show that for any w-$(p,2,M)$-atom a, $M\in\mathbb{N}$, there exists a constant $C>0$ independent of a such that $\|L^{i\gamma}(a)\|_{L^p(w)}\leq C$. Let a be a w-$(p,2,M)$-atom with $supp\,a\subseteq B=B(x_0,r_B)$, $\|a\|_{L^2(B)}\leq|B|^{1/2}w(B)^{-1/p}$. We write

$$\begin{aligned}
\big\|L^{i\gamma}(a)\big\|_{L^p(w)}^p&=\int_{2B}\big|L^{i\gamma}(a)(x)\big|^p w(x)\,dx+\int_{(2B)^c}\big|L^{i\gamma}(a)(x)\big|^p w(x)\,dx\\
&=I_1+I_2.
\end{aligned}$$

Set $s=2/p>1$. Note that $w\in RH_{s'}$, then it follows from Hölder's inequality, the L^2 boundedness of $L^{i\gamma}$ and Lemma 2.1 that

$$\begin{aligned}
I_1&\leq\Big(\int_{2B}\big|L^{i\gamma}(a)(x)\big|^2\,dx\Big)^{p/2}\Big(\int_{2B}w(x)^{s'}\,dx\Big)^{1/s'}\\
&\leq C\|a\|_{L^2(B)}^p\cdot\frac{w(2B)}{|2B|^{1/s}}\\
&\leq C. \qquad (4.1)
\end{aligned}$$

On the other hand, by using Hölder's inequality and the fact that $w \in RH_{s'}$, we can get

$$\begin{aligned} I_2 &= \sum_{k=1}^{\infty} \int_{2^{k+1}B\backslash 2^k B} \big|L^{i\gamma}(a)(x)\big|^p w(x)\,dx \\ &\le C \sum_{k=1}^{\infty} \Big(\int_{2^{k+1}B\backslash 2^k B} \big|L^{i\gamma}(a)(x)\big|^2\,dx \Big)^{p/2} \cdot \frac{w(2^{k+1}B)}{|2^{k+1}B|^{1/s}}. \end{aligned} \tag{4.2}$$

For any $x \in 2^{k+1}B\backslash 2^k B$, $k = 1, 2, \dots$, by the expression (1.2), we can write

$$\begin{aligned} \big|L^{i\gamma}(a)(x)\big| &\le C \int_0^{\infty} e^{-tL}(a)(x)\frac{dt}{t} \\ &\le C \int_0^{r_B^2} e^{-tL}(a)(x)\frac{dt}{t} + C \int_{r_B^2}^{\infty} e^{-tL}(a)(x)\frac{dt}{t} \\ &= \mathrm{I}+\mathrm{II}. \end{aligned}$$

For the term I, we observe that when $x \in 2^{k+1}B\backslash 2^k B$, $y \in B$, then $|x-y| \ge 2^{k-1}r_B$. Hence, by using Hölder's inequality and the estimate (1.1), we deduce

$$\begin{aligned} \big|e^{-tL}a(x)\big| &\le C \cdot \frac{t^{1/2}}{(2^{k-1}r_B)^{n+1}} \int_B \big|a(y)\big|\,dy \\ &\le C \cdot \frac{t^{1/2}}{(2^k r_B)^{n+1}} \|a\|_{L^2(\mathbb{R}^n)} |B|^{1/2} \\ &\le C \cdot w(B)^{-1/p} \frac{t^{1/2}}{2^{k(n+1)} \cdot r_B}. \end{aligned} \tag{4.3}$$

So we have

$$\begin{aligned} \mathrm{I} &\le C \cdot \frac{1}{2^{k(n+1)} w(B)^{1/p}} \cdot \frac{1}{r_B} \int_0^{r_B^2} \frac{dt}{\sqrt{t}} \\ &\le C \cdot \frac{1}{2^{k(n+1)} w(B)^{1/p}}. \end{aligned}$$

We now turn to estimate the other term II. In this case, since there exists a function $b \in \mathcal{D}(L^M)$ such that $a = L^M b$ and $\|b\|_{L^2(B)} \le r_B^{2M} |B|^{1/2} w(B)^{-1/p}$,

then it follows from Hölder's inequality and Lemma 3.1 that

$$
\begin{aligned}
\left|e^{-tL}a(x)\right| &= \left|(tL)^M e^{-tL}b(x)\right| \cdot \frac{1}{t^M} \\
&\le C \cdot \frac{1}{(2^{k-1}r_B)^{n+1}} \int_B |b(y)|\,dy \cdot \frac{1}{t^{M-1/2}} \\
&\le C \cdot \frac{1}{(2^k r_B)^{n+1}} \|b\|_{L^2(\mathbb{R}^n)} |B|^{1/2} \cdot \frac{1}{t^{M-1/2}} \\
&\le C \cdot \frac{r_B^{2M-1}}{2^{k(n+1)}w(B)^{1/p}} \cdot \frac{1}{t^{M-1/2}}. \qquad (4.4)
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\mathrm{II} &\le C \cdot \frac{1}{2^{k(n+1)}w(B)^{1/p}} \cdot r_B^{2M-1} \int_{r_B^2}^{\infty} \frac{dt}{t^{M+1/2}} \\
&\le C \cdot \frac{1}{2^{k(n+1)}w(B)^{1/p}},
\end{aligned}
$$

where in the last inequality we have used the fact that $M \ge 1$. Therefore, by combining the above estimates for I and II, we obtain

$$
\left|L^{i\gamma}(a)(x)\right| \le C \cdot \frac{1}{2^{k(n+1)}w(B)^{1/p}}, \quad \text{when } x \in 2^{k+1}B\backslash 2^k B. \qquad (4.5)
$$

Substituting the above inequality (4.5) into (4,2) and using Lemma 2.1, then we have

$$
\begin{aligned}
I_2 &\le C \sum_{k=1}^{\infty} \frac{1}{2^{kp(n+1)}w(B)} \cdot w(2^{k+1}B) \\
&\le C \sum_{k=1}^{\infty} \frac{1}{2^{k[p(n+1)-n]}} \\
&\le C, \qquad (4.6)
\end{aligned}
$$

where the last series is convergent since $p > n/(n+1)$. Summarizing the estimates (4.1) and (4.6) derived above, we complete the proof of Theorem 1.1. $\square$

Proof of Theorem 1.2. Since the operator $L^{i\gamma}$ is linear and bounded on $L^2(\mathbb{R}^n)$, then by using Theorems 3.3, 3.5 and 3.8, it suffices to verify that for every w-$(p,2,2M)$-atom a, the function $m = L^{i\gamma}(a)$ is a multiple of w-$(p,2,M,\varepsilon)$-molecule for some $\varepsilon > 0$, and the multiple constant is independent of

a. Let a be a w-$(p,2,2M)$-atom with $supp\, a \subseteq B = B(x_0, r_B)$. Then by definition, there exists a function $b \in \mathcal{D}(L^{2M})$ such that $a = L^{2M}(b)$ and $\|(r_B^2 L)^k b\|_{L^2(B)} \le r_B^{4M} |B|^{1/2} w(B)^{-1/p}$, $k = 0, 1, \dots, 2M$. We set $\tilde{b} = L^{i\gamma}(L^M b)$, then we have $m = L^M(\tilde{b})$. Obviously, $m(x) \in L^2(\mathbb{R}^n)$. Moreover, for $k = 0, 1, \dots, M$, we can deduce

$$
\begin{aligned}
\big\|(r_B^2 L)^k \tilde{b}\big\|_{L^2(2B)} &= \frac{1}{r_B^{2M}} \big\|L^{i\gamma}[(r_B^2 L)^{M+k} b]\big\|_{L^2(2B)} \\
&\le C \cdot \frac{1}{r_B^{2M}} \big\|(r_B^2 L)^{M+k} b\big\|_{L^2(B)} \\
&\le C \cdot r_B^{2M} |B|^{1/2} w(B)^{-1/p}.
\end{aligned}
$$

It remains to estimate $\|(r_B^2 L)^k \tilde{b}\|_{L^2(2^{j+1}B \setminus 2^j B)}$ for $k = 0, 1, \dots, M$, $j = 1, 2, \dots$. We write

$$
\begin{aligned}
\big|(r_B^2 L)^k \tilde{b}(x)\big| &= \big|L^{i\gamma}[(r_B^2 L)^k L^M b](x)\big| \\
&\le C \int_0^{r_B^2} e^{-tL}[r_B^{2k} L^{M+k} b](x) \frac{dt}{t} + C \int_{r_B^2}^{\infty} e^{-tL}[r_B^{2k} L^{M+k} b](x) \frac{dt}{t} \\
&= \mathrm{I}' + \mathrm{II}'.
\end{aligned}
$$

As mentioned in the proof of Theorem 1.1, we know that when $x \in 2^{j+1}B \setminus 2^j B$, $y \in B$, then $|x-y| \ge 2^{j-1} r_B$, $j = 1, 2, \dots$. It follows from Hölder's inequality and the estimate (1.1) that

$$
\begin{aligned}
\mathrm{I}' &\le C \int_0^{r_B^2} \frac{t^{1/2}}{(2^{j-1} r_B)^{n+1}} \big\|r_B^{2k} L^{M+k} b\big\|_{L^2(\mathbb{R}^n)} |B|^{1/2} \frac{dt}{t} \\
&\le C \cdot \frac{1}{(2^j r_B)^{n+1}} \Big(\frac{1}{r_B^{2M}} \cdot r_B^{4M} |B|^{1/2} w(B)^{-1/p} \Big) |B|^{1/2} \int_0^{r_B^2} \frac{dt}{\sqrt{t}} \\
&\le C \cdot \frac{1}{2^{j(n+1)}} \cdot r_B^{2M} w(B)^{-1/p}. \qquad (4.7)
\end{aligned}
$$

Since $B \subseteq 2^j B$, $j = 1, 2, \dots$, then by using Lemma 2.2, we can get

$$
\frac{w(B)}{w(2^j B)} \ge C \cdot \frac{|B|}{|2^j B|}. \qquad (4.8)
$$

Hence

$$
\mathrm{I}' \le C \cdot \frac{1}{2^{j[(n+1)-n/p]}} \cdot r_B^{2M} w(2^j B)^{-1/p}, \quad \text{when } x \in 2^{j+1}B \setminus 2^j B.
$$

Applying Hölder's inequality and Lemma 3.1, we obtain

$$\begin{aligned}
\mathrm{II}' &\le C\cdot r_B^{2k}\int_{r_B^2}^{\infty}(tL)^{M+k}e^{-tL}(b)(x)\frac{dt}{t^{M+k+1}}\\
&\le C\cdot r_B^{2k}\int_{r_B^2}^{\infty}\frac{1}{(2^{j-1}r_B)^{n+1}}\|b\|_{L^2(\mathbb{R}^n)}|B|^{1/2}\frac{dt}{t^{M+k+1/2}}\\
&\le C\cdot\frac{1}{(2^j r_B)^{n+1}}\Big(r_B^{4M+2k}|B|w(B)^{-1/p}\Big)\int_{r_B^2}^{\infty}\frac{dt}{t^{M+k+1/2}}\\
&\le C\cdot\frac{1}{2^{j(n+1)}}\cdot r_B^{2M}w(B)^{-1/p}. \qquad (4.9)
\end{aligned}$$

It follows immediately from the above inequality (4.8) that

$$\mathrm{II}'\le C\cdot\frac{1}{2^{j[(n+1)-n/p]}}\cdot r_B^{2M}w(2^jB)^{-1/p},\quad \text{when } x\in 2^{j+1}B\backslash 2^jB.$$

Combining the above estimates for I' and II', we thus obtain

$$\big\|(r_B^2L)^k\tilde{b}\big\|_{L^2(2^{j+1}B\backslash 2^jB)}\le C\cdot\frac{1}{2^{j[(n+1)-n/p]}}\cdot r_B^{2M}|2^jB|^{1/2}w(2^jB)^{-1/p}.$$

Observe that $p>n/(n+1)$. If we set $\varepsilon=(n+1)-n/p$, then we have $\varepsilon>0$. Therefore, we have proved that the function $m=L^{i\gamma}(a)$ is a multiple of w-$(p,2,M,\varepsilon)$-molecule. This completes the proof of Theorem 1.2. □

5 Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. As in the proof of Theorem 1.1, it suffices to prove that for every w-$(p,2,M)$-atom a, $M>(3n)/4$, there exists a constant $C>0$ independent of a such that $\|L^{-\alpha/2}(a)\|_{L^q(w^{q/p})}\le C$. We write

$$\begin{aligned}
\big\|L^{-\alpha/2}(a)\big\|_{L^q(w^{q/p})}^q &= \int_{2B}\big|L^{-\alpha/2}(a)(x)\big|^q w(x)^{q/p}\,dx+\int_{(2B)^c}\big|L^{-\alpha/2}(a)(x)\big|^q w(x)^{q/p}\,dx\\
&= J_1+J_2.
\end{aligned}$$

First note that $0<\alpha<n/2$, $1/q=1/p-\alpha/n$, then we are able to choose a number $\mu>q$ such that $1/\mu=1/2-\alpha/n$. Set $s=2/p$, then by a simple calculation, we can easily see that $(q/p)\cdot(\mu/q)'=s'$ and $1-q/\mu=q/(ps')$. Applying Hölder's inequality, the L^2-L^μ boundedness of $L^{-\alpha/2}$, Lemma 2.1

and $w \in RH_{s'}$, we can get

$$\begin{aligned} J_1 &\le \Big(\int_{2B} \big|L^{-\alpha/2}(a)(x)\big|^{q\cdot\frac{\mu}{q}}\,dx\Big)^{q/\mu}\Big(\int_{2B} w(x)^{\frac{q}{p}\cdot(\frac{\mu}{q})'}\,dx\Big)^{1-q/\mu} \\ &= \Big(\int_{2B} \big|L^{-\alpha/2}(a)(x)\big|^{\mu}\,dx\Big)^{q/\mu}\Big(\int_{2B} w(x)^{s'}\,dx\Big)^{q/(ps')} \\ &\le C\|a\|^q_{L^2(\mathbb{R}^n)}\left(\frac{w(2B)}{|2B|^{1/s}}\right)^{q/p} \\ &\le C. \end{aligned} \tag{5.1}$$

We now turn to deal with J_2. Using the condition $w \in RH_{s'}$ and Hölder's inequality, we obtain

$$\begin{aligned} J_2 &= \sum_{k=1}^{\infty}\int_{2^{k+1}B\backslash 2^kB} \big|L^{-\alpha/2}(a)(x)\big|^q w(x)^{q/p}\,dx \\ &\le C\sum_{k=1}^{\infty}\Big(\int_{2^{k+1}B\backslash 2^kB} \big|L^{-\alpha/2}(a)(x)\big|^{\mu}\,dx\Big)^{q/\mu}\cdot\left(\frac{w(2^{k+1}B)}{|2^{k+1}B|^{1/s}}\right)^{q/p}. \end{aligned} \tag{5.2}$$

For any $x \in 2^{k+1}B\backslash 2^kB$, $k = 1, 2, \dots$, by the expression (1.3), we can write

$$\begin{aligned} \big|L^{-\alpha/2}(a)(x)\big| &\le C\int_0^\infty e^{-tL}(a)(x)\frac{dt}{t^{1-\alpha/2}} \\ &\le C\int_0^{r_B^2} e^{-tL}(a)(x)\frac{dt}{t^{1-\alpha/2}} + C\int_{r_B^2}^\infty e^{-tL}(a)(x)\frac{dt}{t^{1-\alpha/2}} \\ &= \mathrm{III}+\mathrm{IV}. \end{aligned}$$

For the term III, it follows immediately from (4.3) that

$$\begin{aligned} \mathrm{III} &\le C\cdot\frac{1}{2^{k(n+1)}w(B)^{1/p}}\cdot\frac{1}{r_B}\int_0^{r_B^2}\frac{dt}{t^{1/2-\alpha/2}} \\ &\le C\cdot\frac{r_B^\alpha}{2^{k(n+1)}w(B)^{1/p}}. \end{aligned}$$

For the other term IV, by the previous estimate (4.4), we thus have

$$\begin{aligned} \mathrm{IV} &\le C\cdot\frac{1}{2^{k(n+1)}w(B)^{1/p}}\cdot r_B^{2M-1}\int_{r_B^2}^\infty\frac{dt}{t^{M+1/2-\alpha/2}} \\ &\le C\cdot\frac{r_B^\alpha}{2^{k(n+1)}w(B)^{1/p}}, \end{aligned}$$

where the last inequality holds since $M > (3n)/4 > 1/2 + \alpha/2$. Combining the above estimates for III and IV, we obtain

$$\left|L^{-\alpha/2}(a)(x)\right| \le C \cdot \frac{r_B^\alpha}{2^{k(n+1)}w(B)^{1/p}}, \quad \text{when } x \in 2^{k+1}B\backslash 2^kB. \tag{5.3}$$

Substituting the above inequality (5.3) into (5,2) and using Lemma 2.1, we can get

$$\begin{aligned} J_2 &\le C\sum_{k=1}^\infty \left|2^{k+1}B\right|^{q/\mu} \cdot \left(\frac{r_B^\alpha}{2^{k(n+1)}w(B)^{1/p}}\right)^q \cdot \left(\frac{w(2^{k+1}B)}{|2^{k+1}B|^{1/s}}\right)^{q/p} \\ &\le C\sum_{k=1}^\infty \frac{1}{2^{k[q(n+1)-n]}} \\ &\le C, \end{aligned} \tag{5.4}$$

where in the last inequality we have used the fact that $q > p > n/(n+1)$. Therefore, by combining the above inequality (5.4) with (5.1), we conclude the proof of Theorem 1.3. □

Proof of Theorem 1.4. As in the proof of Theorem 1.2, it is enough to show that for every w-$(p,2,2M)$-atom a, the function $m = L^{-\alpha/2}(a)$ is a multiple of w-$(p,2,M,\varepsilon)$-molecule for some $\varepsilon > 0$, and the multiple constant is independent of a. Let a be a w-$(p,2,2M)$-atom with $supp\, a \subseteq B = B(x_0, r_B)$, and $a = L^{2M}(b)$, where $M > (3n)/4 > \max\{\frac{n}{2}(\frac{1}{p}-\frac{1}{2}), \frac{1}{2}+\frac{\alpha}{2}\}$, $b \in \mathcal{D}(L^{2M})$. Set $\tilde{b} = L^{-\alpha/2}(L^M b)$, then we have $m = L^M(\tilde{b})$. It is easy to check that $m(x) \in L^2(\mathbb{R}^n)$. As before, since $0 < \alpha < n/2$, then we may choose a number $\mu > 2$ such that $1/\mu = 1/2 - \alpha/n$. For $k = 0, 1, \dots, M$, by using Hölder's inequality, the L^2-L^μ boundedness of $L^{-\alpha/2}$, we obtain

$$\begin{aligned} \left\|(r_B^2L)^k\tilde{b}\right\|_{L^2(2B)} &\le \frac{1}{r_B^{2M}}\left\|L^{-\alpha/2}[(r_B^2L)^{M+k}b]\right\|_{L^\mu(2B)}|2B|^{1/2-1/\mu} \\ &\le C \cdot \frac{1}{r_B^{2M}}\left\|(r_B^2L)^{M+k}b\right\|_{L^2(B)}|B|^{1/2-1/\mu} \\ &\le C \cdot r_B^{2M}|B|^{1/2+\alpha/n}w(B)^{-1/p}. \end{aligned} \tag{5.5}$$

Note that $1/q = 1/p - \alpha/n$, then a straightforward computation yields that $q/p < (2/p)'$ whenever $0 < \alpha < n/2$. By our assumption $w \in RH_{(2/p)'}$, then we have $w \in RH_{q/p}$. Consequently

$$w^{q/p}(B)^{p/q} \le C \cdot \frac{w(B)}{|B|^{1-p/q}},$$

which implies

$$w(B)^{-1/p} \le C \cdot |B|^{1/q-1/p} w^{q/p}(B)^{-1/q}. \tag{5.6}$$

Substituting the above inequality (5.6) into (5.5), we can get

$$\big\|(r_B^2 L)^k \tilde{b}\big\|_{L^2(2B)} \le C \cdot r_B^{2M} |B|^{1/2} w^{q/p}(B)^{-1/q}. \tag{5.7}$$

It remains to estimate $\|(r_B^2 L)^k \tilde{b}\|_{L^2(2^{j+1}B \backslash 2^j B)}$ for $k = 0, 1, \dots, M$, $j = 1, 2, \dots$. We write

$$\begin{aligned}
&\big|(r_B^2 L)^k \tilde{b}(x)\big| \\
=&\big|L^{-\alpha/2}[(r_B^2 L)^k L^M b](x)\big| \\
\le& C \int_0^{r_B^2} e^{-tL}[r_B^{2k} L^{M+k} b](x) \frac{dt}{t^{1-\alpha/2}} + C \int_{r_B^2}^{\infty} e^{-tL}[r_B^{2k} L^{M+k} b](x) \frac{dt}{t^{1-\alpha/2}} \\
=&\mathrm{III}' + \mathrm{IV}'.
\end{aligned}$$

For the term III', by using the same arguments as in the proof of (4.7), we have

$$\begin{aligned}
\mathrm{III}' &\le C \cdot \frac{1}{2^{j(n+1)}} \cdot r_B^{2M} w(B)^{-1/p} \frac{1}{r_B} \int_0^{r_B^2} \frac{dt}{t^{1/2-\alpha/2}} \\
&\le C \cdot \frac{1}{2^{j(n+1)}} \cdot r_B^{2M+\alpha} w(B)^{-1/p}.
\end{aligned}$$

For the term IV', we follow the same arguments as that of (4.9) and then obtain

$$\begin{aligned}
\mathrm{IV}' &\le C \cdot \frac{1}{2^{j(n+1)}} \cdot w(B)^{-1/p} r_B^{4M+2k-1} \int_{r_B^2}^{\infty} \frac{dt}{t^{M+k+1/2-\alpha/2}} \\
&\le C \cdot \frac{1}{2^{j(n+1)}} \cdot r_B^{2M+\alpha} w(B)^{-1/p}.
\end{aligned}$$

Combining the above estimates for III' and IV', we can get

$$\big|(r_B^2 L)^k \tilde{b}(x)\big| \le C \cdot \frac{1}{2^{j(n+1)}} \cdot r_B^{2M+\alpha} w(B)^{-1/p}, \quad \text{when } x \in 2^{j+1}B \backslash 2^j B.$$

Since $w \in A_1$, then it follows from the previous inequality (4.8) that

$$\big|(r_B^2 L)^k \tilde{b}(x)\big| \le C \cdot \frac{1}{2^{j[(n+1)-n/p]}} \cdot r_B^{2M+\alpha} w(2^j B)^{-1/p}, \quad \text{when } x \in 2^{j+1}B \backslash 2^j B.$$

Similar to the proof of (5.6), we can also show that

$$w(2^j B)^{-1/p} \le C \cdot |2^j B|^{1/q-1/p} w^{q/p}(2^j B)^{-1/q}.$$

Hence

$$\left|(r_B^2 L)^k \tilde{b}(x)\right| \leq C \cdot \frac{1}{2^{j[(n+1)-n/q]}} \cdot r_B^{2M} w^{q/p}(2^j B)^{-1/q}, \quad \text{when } x \in 2^{j+1}B \backslash 2^j B.$$

Therefore

$$\left\|(r_B^2 L)^k \tilde{b}\right\|_{L^2(2^{j+1}B\backslash 2^j B)} \leq C \cdot \frac{1}{2^{j[(n+1)-n/q]}} \cdot r_B^{2M} |2^j B|^{1/2} w^{q/p}(2^j B)^{-1/q}. \tag{5.8}$$

Observe that $1 \geq q > p > n/(n+1)$. If we set $\varepsilon = (n+1) - n/q$, then $\varepsilon > 0$. Summarizing the estimates (5.7) and (5.8) derived above, we finally conclude the proof of Theorem 1.4. □

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